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5.5 The Real Zeros of Polynomials

* **Remainder Theorem**: If $f(x)$ is a poly and is divided by $x-c$, then the remainder is $f(c)$.

ex. 1) Find Remainder when $f(x) = x^3 - 4x^2 - 5$ is divided by:

a) $x-3 \rightarrow x=3$ so $f(3) = (3)^3 - 4(3)^2 - 5 = \boxed{-14}$

b) $x+2 \rightarrow x=-2$ so $f(-2) = (-2)^3 - 4(-2)^2 - 5 = \boxed{-29}$

* **Factor Theorem**: if f is a poly. Then $x-c$ is a factor of $f(x)$ if and only if $f(c)=0$

so... if $f(c)=0$ then $x-c$ is a factor of f
if $x-c$ is a factor of f then $f(c)=0$

ex. 2) Determine whether $f(x) = 2x^3 - x^2 + 2x - 3$ has the factor

(a) $x-1 \rightarrow x=1$ $f(1) = 2(1)^3 - (1)^2 + 2(1) - 3 = 0$

since $\boxed{f(1)=0, \ x-1 \text{ is a factor of } f(x)}$

(b) $x+2 \rightarrow x=-2$ $f(-2) = -27$

since $\boxed{f(-2) \neq 0, \ x+2 \text{ is not a factor of } f(x)}$

* **Theorem for # of real zeros**: A polynomial cannot have more real zeros than its degree.

*Descartes' Rule of Signs: If n is the number of sign changes in $P(x)$ or $P(-x)$, then the # of positive or negative roots may equal n or, $n-2$, or $n-4$, or $n-6$, etc....

ex.) • if $P(x)$ has $n=7$ number of sign changes, the possible number of positive real zeros are:

7, 5, 3, or 1

• if $P(-x)$ has $n=6$ number of sign changes, the possible number of negative real zeros are:

6, 4, 2, or 0

ex.3) Possible real zeros for $f(x)=3x^7-4x^4+3x^3+2x^2-x-3$

degree 7 = can have at most 7 real zeros

$$f(x)=3x^7-4x^4+3x^3+2x^2-x-3$$

$+\to-$ $-\to+$ $+\to-$

$\boxed{\text{f(x) has 3 sign changes which means there are either 3 real positive zeros or 1.}}$

$$f(-x)=3(-x)^7-4(-x)^4+3(-x)^3+2(-x)^2-(-x)-3$$
$$=-3x^7-4x^4-3x^3+2x^2+x-3$$

$-\to+$ $+\to-$

$\boxed{\text{f(-x) has 2 sign changes so f(x) has either 2 or 0 negative real zeros}}$

* <u>Theorem Rational zeros</u>: can only be used when coefficients are integers

in the form:
$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$$

where $a_n \neq 0$ and $a_0 \neq 0$. The possible rational zeros of f are $\frac{p}{q}$ in lowest terms where p are factors of a_0 and q are factors of a_n

ex. 4) List possible rational zeros of:
$$f(x) = \underset{a_n}{2}x^3 + 11x^2 - 7x \overset{a_0}{-6}$$
q are factors of 2   p are factors of -6

① List all integer factors.

p: ±1, ±2, ±3, ±6 ← factors of −6

q: ±1, ±2 ← factors of 2

② Form all possible ratios of $\frac{p}{q}$.

$\frac{p}{q}$: $\pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{3}{1}, \pm\frac{3}{2}, \pm\frac{6}{1}, \pm\frac{6}{2}$

③ simplify: $\pm1, \pm\frac{1}{2}, \pm2, \pm3, \pm\frac{3}{2}, \pm6$

④ $\boxed{\text{so if } f(x) \text{ has a rational zero it will be in the list which has 12 possibilities}}$

Find the Real zeros of a Poly

ex 5) $f(x)=2x^3+11x^2-7x-6$

Step 1: Determine Max # of zeros

$\boxed{\text{Degree 3 = at most 3 real zeros}}$

Step 2: Use Descartes' Rule to find possible +/- zeros

$f(x)=2x^3+11x^2-7x-6$ $\boxed{\text{1 positive real zero}}$

$f(-x)=-2x^3+11x^2+7x-6$ $\boxed{\text{2 or 0 negative real zeros}}$

Step 3: List potential rational zeros

$a_0=-6$ $a_n=2$

$p: \pm 1, \pm 2, \pm 3, \pm 6$

$q: \pm 1, \pm 2$

$\frac{p}{q}: \pm\frac{1}{1}, \pm\frac{1}{2}, \pm\frac{2}{1}, \pm\frac{2}{2}, \pm\frac{3}{1}, \pm\frac{3}{2}$

$\pm\frac{6}{1}, \pm\frac{6}{2}$

so $\boxed{\pm 1, \pm 2, \pm 3, \pm 6, \pm\frac{1}{2}, \pm\frac{3}{2}}$

Step 4: Test the rational zeros to determine if it is a zero of f(x).

$\boxed{f(1)=0}$ $f(2)=40$ $f(3)=126$ $f(6)=780$ ....etc

$f(-1)=10$ $f(-2)=36$ $f(-3)=60$ $\boxed{f(-6)=0}$

Step 5: Use synthetic division to simplify/factor the poly

$$\begin{array}{r|rrrr} 1 & 2 & 11 & -7 & -6 \\ & \downarrow & 2 & 13 & 6 \\ \hline & 2 & 13 & 6 & 0 \end{array}$$

so $(x-1)(2x^2+13x+6)$

$$\begin{array}{r|rrr} -6 & 2 & 13 & 6 \\ & \downarrow & -12 & -6 \\ \hline & 2 & 1 & 0 \end{array}$$

$(x-1)(x+6)(2x+1)$

$\boxed{\text{zeros are } 1, -6, -\frac{1}{2}}$

ex.6) $f(x)=x^5-7x^4+19x^3-37x^2+60x-36$

① at most 5 real zeros

② $\boxed{\text{5, 3, or 1 positive real zeros}}$

$f(-x) = -x^5-7x^4-19x^3-37x^2-60-36$

$\boxed{\text{no negative zeros}}$

③ (no negatives) p: $\boxed{1,2,3,4,6,9,12,18,36}$

q: 1

④ check them!

$f(1)=0$

$f(3)=0$

so

$$\begin{array}{r|rrrrrr} 1 & 1 & -7 & 19 & -37 & 60 & -36 \\ & \downarrow & 1 & -6 & 13 & -24 & 36 \\ \hline & 1 & -6 & 13 & -24 & 36 & 0 \end{array}$$

$(x-1)(x^4-6x^3+13x^2-24x+36)$

$$\begin{array}{r|rrrrr} 3 & 1 & -6 & 13 & -24 & 36 \\ & \downarrow & 3 & -9 & 12 & -36 \\ \hline & 1 & -3 & 4 & -12 & 0 \end{array}$$

$(x-1)(x-3)\underbrace{(x^3-3x^2+4x-12)}_{\text{factor}}$

	x	-3
x^2	x^3	$-3x^2$
4	$4x$	-12

$(x-1)(x-3)(x^2+4)(x-3)$

$\boxed{\text{real zeros: 1 and 3} \leftarrow \text{repeated 2 times}}$

Solving a poly

ex.7) Find real solutions of: $x^5 - 7x^4 + 19x^3 - 37x^2 + 60x - 36 = 0$

$\boxed{\text{The real solutions are the real zeros so } 1 \text{ and } 3}$

* **Bounds on zeros Theorem**: Let f be a poly with a leading coefficient that is positive:

- If $M > 0$ is a real # & if the answer by division of f by $x - M$ contains only #s that are positive or zero, then M is an upper bound to the zeros of f
- If $m < 0$ is a real # & if the answer by division of f by $x - m$ contains #s that alternate positive (or 0) & negative (or 0) then m is a lower bound to the zeros of f.

M = upper bound if no zero is greater than M

m = lower bound if no zero of f is less than m

$m \leq \text{any zero of } f \leq M$

ex.8) Find upper | lower bounds for $f(x) = 2x^3 + 11x^2 - 7x - 6$

① List potential rational zeros: $\pm 1, \pm 2, \pm 3, \pm 6, \pm \frac{1}{2}, \pm \frac{3}{2}$

② To find upper → check the smallest possible positive integer

③ To find lower → check the largest negative integer

☆ Check by synthetic division to see if it meets the theorem

synthetic division summary

r	coefficients of $g(x)$			Remainder
1	2	13	6	0 ← all positive = upper bound
−1	2	9	−16	10
−2	2	7	−21	36
−3	2	5	−22	60
−6	2	−1	−1	0
−7	2	−3	14	−104

alternate signs = lower bound

$\boxed{\text{upper bound} = 1 \quad \text{lower bound} = -7}$

* <u>intermediate value Theorem</u>: Let f denote a poly. If $a < b$ and if $f(a)$ & $f(b)$ are of opposite sign, there is at least one real zero of f between a & b.

ex. 9) show $f(x) = x^5 - x^3 - 1$ has a zero between 1 & 2

$f(1) = -1$ and $f(2) = 23$

$\boxed{\text{opposite signs so by the IVT it has at least 1 zero between 1 \& 2}}$