13.5 Binomial Theorem

$\binom{n}{j}$ reads "n taken j at a time"

if j and n are integers w/ $0 \le j \le n$

$$\binom{n}{j} = \frac{n!}{j!(n-j)!}$$

ex.1) evaluate: (a) $\binom{3}{1} = \frac{3!}{1!(3-1)!} = \frac{3!}{1!2!} = \frac{3\cdot 2\cdot 1}{1\cdot(2\cdot 1)} = \frac{6}{2} = \boxed{3}$

(b) $\binom{8}{7} = \frac{8!}{7!1!} = \frac{8\cdot \cancel{7!}}{\cancel{7!}\cdot 1!} = \frac{8}{1} = \boxed{8}$

Binomial Theorem

let x & a be $\mathbb{R}$, for any + integer n, we have

$$\boxed{\begin{aligned}(x+a)^n &= \binom{n}{0}x^n + \binom{n}{1}ax^{n-1} + \dots + \binom{n}{j}a^j x^{n-j} + \dots + \binom{n}{n}a^n \\ &= \sum_{j=0}^{n} \binom{n}{j} x^{n-j} a^j\end{aligned}}$$

or use the

PASCAL TRIANGLE

use to find the coefficients instead of $\binom{n}{j}$

$n=0 \rightarrow$							1						
$n=1 \rightarrow$						1		1					
$n=2 \rightarrow$					1		2		1				
$n=3 \rightarrow$				1		3		3		1			
$n=4 \rightarrow$			1		4		6		4		1		
$n=5 \rightarrow$		1		5		10		10		5		1	
$n=6 \rightarrow$	1		6		15		20		15		6		1
	⋮	⋮	⋮		⋮		⋮		⋮		⋮		⋮

ex.2) Expand $(x+\underset{\substack{\uparrow \\ a}}{2})^{5 \leftarrow n}$:

① use row $n=5 \rightarrow 1, 5, 10, 10, 5, 1$

*exponent on **a** starts at **0** and goes to **n**

*exponent on **x** starts at **n** and goes to **0**

② $1(\underset{a}{2})^0(\underset{x}{x})^5 + 5\cdot(\underset{a}{2})^1(\underset{x}{x})^4 + 10\cdot(\underset{a}{2})^2(\underset{x}{x})^3 + \ldots$

$\ldots + 10(\underset{a}{2})^3(\underset{x}{x})^2 + 5(\underset{a}{2})^4(\underset{x}{x})^1 + 1(\underset{a}{2})^5(\underset{x}{x})^0$

$= \boxed{x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32}$

ex.3) Expand $(\underset{\substack{\uparrow \\ x}}{\underbrace{2y}} - \underset{\substack{\uparrow \\ a}}{3})^{4 \leftarrow n}$:

① use row $n=4 \rightarrow 1, 4, 6, 4, 1$

②

$1(-3)^0(2y)^4 + 4(-3)^1(2y)^3 + 6(-3)^2(2y)^2 + 4(-3)^3(2y)^1 + 1(-3)^4(2y)^0$

$= \boxed{16y^4 - 96y^3 + 216y^2 - 216y + 81}$

ex.4) Find the coefficient of y^8 in the expansion of $(2y+3)^{10}$

① row $n=10 \rightarrow 1, 10, 45, 120, 210, 252, 210, 120, 45, 10, 1$

②

$1(3)^0(2y)^{10} + 10(3)^1(2y)^9 + 45(3)^2(2y)^8 + \ldots\ldots$

$45(9)(256y^8) = \boxed{103{,}680y^8}$

Formula to find term containing x^j:

$$\binom{n}{n-j} a^{n-j} x^j$$

ex. 5) Find the 6th term of $(x+2)^9$

x^9 x^8 x^7 x^6 x^5 x^4 ...
1st 2nd 3rd 4th 5th 6th ← looking for x^4

$j=4$ $n=9$ $a=2$

1st method:

$$\binom{9}{9-4}(2)^{9-4}x^4 = \binom{9}{5}(2)^5 x^4$$

$$\frac{9!}{5!\,4!}(32)x^4 = \boxed{4{,}032x^4}$$

2nd method:

row $n=9$ → 1, 9, 36, 84, 126, $\underline{126}$, 84, 36, 9, 1 (6th term)

$$1(2)^0(x)^9 + 9(2)^1(x)^8 + 36(2)^2(x)^7 + 84(2)^3(x)^6$$

$$+126(2)^4(x)^5 + \boxed{126(2)^5(x)^4}$$

↑ 6th term

$$126(32)(x)^4 = \boxed{4{,}032x^4}$$