12.5 Partial Fraction Decomposition

★ Consider adding 2 rational expressions:

$$\frac{3}{x+4} + \frac{2}{x-3} = \frac{2(x+4)+3(x-3)}{(x+4)(x-3)} = \frac{5x-1}{x^2+x-12}$$

the reverse procedure starting with the expression $\frac{5x-1}{x^2+x-12}$ and writing it as the sum of 2 simpler fractions $\frac{3}{x+4}$ and $\frac{2}{x-3}$ is called <u>partial fraction decomposition</u> where $\frac{3}{x+4}$ and $\frac{2}{x-3}$ are called <u>partial fractions</u>.

★ The partial fraction decomposition $\frac{P}{Q}$ in lowest terms means Q will contain factors of one or both types:

- Linear factors of the form $x-a$, where a is a real #.
- irreducible quadratic factors of the form ax^2+bx+c, where $a,b,c \in \mathbb{R}$ and $b^2-4ac<0$.

Case 1: Q has only nonrepeated linear factors.

so $Q(x) = (x-a_1)(x-a_2)\cdots(x-a_n)$
where a_1, a_2, a_n etc are <u>not</u> equal

so $$\frac{P(x)}{Q(x)} = \frac{A_1}{x-a_1} + \frac{A_2}{x-a_2} + \cdots + \frac{A_n}{x-a_n}$$

where $A_1, A_2, \cdots A_n$ are to be determined ex.1) Find PFD of $\dfrac{x}{x^2-5x+6}$

Step 1: Factor the denominator $\quad x^2-5x+6=(x-2)(x-3)$

Step 2: Decompose the rational $\quad \left(\dfrac{x}{x^2-5x+6}\right)(x-2)(x-3)=\left(\dfrac{A}{x-2}+\dfrac{B}{x-3}\right)(x-2)(x-3)$

Step 3: Clear the fraction by multiplying by denom. on both sides

$$x=A(x-3)+B(x-2)$$

$$0\cancel{x}=Ax-3A+Bx-2B$$

Put x's together

$$x=Ax+Bx-3A-2B$$

Factor out x's

$$x=x(A+B)+(-3A-2B)$$

make equations for your x's & constants

Step 4: For equation to be true ($x=x$) A+B must be 1 and -3A-2B must be 0

$$\begin{cases} A+B=1 \\ -3A-2B=0 \end{cases}$$

Step 5: Solve system of equations (use any method)

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ -3 & -2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & 3 \end{array}\right]$$

$R_2=3r_1+r_2$

$A+B=1$ & $\boxed{B=3 \text{ so } A=-2}$

Step 6: Write PFD using equation in step 2

$$\boxed{\frac{x}{x^2-5x+6}=\frac{-2}{x-2}+\frac{3}{x-3}}$$

← Try #13 on pg. 908 ⟶

Case 2: Q has repeated linear factors

If Q has repeated factors, say $(x-a)^n$, $n \geq 2$ then the PFD of $\frac{P}{Q}$ is

$$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \dots + \frac{A_n}{(x-a)^n}$$

where $A_1, A_2 \cdots A_n$ are to be determined

ex.2) Find the PFD of $\dfrac{x+2}{x^3-2x^2+x}$

Step 1: Factor denom. $x^3-2x^2+x = x(x^2-2x+1)$
$= x(x-1)^2$

Step 2: Write PFD equation using cases

$$\frac{x+2}{x^3-2x^2+x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}$$

multiply each side by $x(x-1)^2$

Step 3: Clear fractions $x+2 = A(x-1)^2 + Bx(x-1) + Cx$

Step 4: Get rid of the parenthesis

$x+2 = A(x^2-2x+1) + B(x^2-x) + Cx$

$x+2 = Ax^2 - 2Ax + 1A + Bx^2 - Bx + Cx$

Step 5: Write 3 equations for all terms

$$\begin{aligned} x^2 &\rightarrow \\ x &\rightarrow \\ c &\rightarrow \end{aligned} \begin{cases} 0 = A + B \\ 1 = -2A - B + C \\ \underline{2 = A} \end{cases}$$

Step 6: Solve the system of equations using any method

Substitute:

$0 = 2 + B \rightarrow \underline{B = -2}$

$1 = -2(2) - (-2) + C$

$1 = -4 + 2 + C$

$1 = -2 + C \rightarrow \underline{C = 3}$

$\longrightarrow$

Step 7: write answer

$$\boxed{\frac{x+2}{x^3-2x^2+x} = \frac{2}{x} + \frac{-2}{x-1} + \frac{3}{(x-1)^2}}$$

ex.3) Find PFD of $\frac{x^3-8}{x^2(x-1)^3}$

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} + \frac{D}{(x-1)^2} + \frac{E}{(x-1)^3}$$

→ clear fractions:

$$x^3-8 = Ax(x-1)^3 + B(x-1)^3 + Cx^2(x-1)^2 + Dx^2(x-1) + Ex^2$$

→ get rid of parenthesis:

$$x^3-8 = Ax(x^3-3x^2+3x-1) + B(x^3-3x^2+3x-1) + Cx^2(x^2-2x+1) + D(x^3-x^2) + Ex^2$$

$$x^3-8 = A(x^4-3x^3+3x^2-1x) + B(x^3-3x^2+3x-1) + C(x^4-2x^3+x^2) + D(x^3-x^2) + Ex^2$$

→ create equations

$x^4 \rightarrow$ $0 = A + C$

$x^3 \rightarrow$ $1 = -3A + B - 2C + D$

$x^2 \rightarrow$ $0 = 3A - 3B + C - D + E$

$x \rightarrow$ $0 = -A + 3B$

$c \rightarrow$ $-8 = -B$

→ substitute to get

$A = 24$

$B = 8$

$C = -24$

$D = 17$

$E = -7$

→ Try #19 after ex.3

ex.3 answer:

$$\frac{x^3-8}{x^2(x-1)^3} = \frac{24}{x} + \frac{8}{x^2} + \frac{-24}{x-1} + \frac{17}{(x-1)^2} + \frac{-7}{(x-1)^3}$$

case 3: Q contains a nonrepeated irreducible quadratic factor then the PFD is:

$$\frac{Ax+B}{ax^2+bx+c}$$

ex.4) Find the PFD of $\frac{3x-5}{(x-1)(x^2+x+1)}$

$$(x-1)(x^2+x+1)\left(\frac{3x-5}{(x-1)(x^2+x+1)}\right) = \left(\frac{A}{x-1} + \frac{Bx+C}{x^2+x+1}\right)(x-1)(x^2+x+1)$$

$$3x-5 = A(x^2+x+1)+(Bx+C)(x-1)$$

$$3x-5 = Ax^2+Ax+A+Bx^2-Bx+Cx-C$$

$x^2 \to \quad 0 = A+B$

$x \to \quad 3 = A-B+C$

$C \to \quad -5 = A \quad -C$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 3 \\ 1 & 0 & -1 & -5 \end{array}\right] \xrightarrow[R_3 = -1r_1+r_3]{R_2 = -1r_1+r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 3 \\ 0 & -1 & -1 & -5 \end{array}\right] \xrightarrow{R_2 = -3r_3+r_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 18 \\ 0 & -1 & -1 & -5 \end{array}\right] \xrightarrow{R_3 = r_2+r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 3 & 13 \end{array}\right]$$

$$\xrightarrow{R_3 = \frac{1}{3}r_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 4 & 18 \\ 0 & 0 & 1 & \frac{13}{3} \end{array}\right]$$

$A+B=0 \to A = -\frac{2}{3}$

$B+4C=18 \to B = 18-\frac{52}{3} = \frac{54}{3}-\frac{52}{3} = \frac{2}{3}$

$C = 13/3$

$$\boxed{\frac{3x-5}{(x-1)(x^2+x+1)} = \frac{-2/3}{x-1} + \frac{2/3x+13/3}{x^2+x+1}}$$

→ Try #21 after ex.4

case 4: Q contains repeated irreducible quad. factor:

then:

$$\frac{A_1x+B_1}{ax^2+bx+c}+\frac{A_2x+B_2}{(ax^2+bx+c)^2}+\cdots+\frac{A_nx+B_n}{(ax^2+bx+c)^n}$$

ex.5) Find the PFD of $\frac{x^3+x^2}{(x^2+4)^2}=\frac{Ax+B}{x^2+4}+\frac{Cx+D}{(x^2+4)^2}$

$x^3+x^2=(Ax+B)(x^2+4)+(Cx+D)$

$x^3+x^2=Ax^3+4Ax+Bx^2+4B+Cx+D$

$x^3 \rightarrow 1=A$

$x^2 \rightarrow 1=B$

$x \rightarrow 0=4A+C$

$c \rightarrow 0=4B+D$

so if $A=1$ & $B=1$, then $C=-4$ and $D=-4$

$$\boxed{\frac{x^3+x^2}{(x^2+4)^2}=\frac{x+1}{x^2+4}+\frac{-4x-4}{(x^2+4)^2}}$$

→ Try #35

HW: pg. 908 #5-43 odd