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12.4 Matrix Algebra

examples of matrices:

$2\times2 \rightarrow \begin{bmatrix} 5 & 0 \\ -6 & 1 \end{bmatrix}$ $\qquad 1\times3 \rightarrow \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$

rows, columns

$2\times3 \rightarrow \begin{bmatrix} 6 & -2 & 4 \\ 4 & 3 & 5 \end{bmatrix}$

* Equal matrices: $A=B$ if A & B have same # of rows & colum s & each entry a_{ij} is equal to b_{ij}.

ex. $\begin{bmatrix} 2 & 1 \\ 0.5 & -1 \end{bmatrix} = \begin{bmatrix} \sqrt{4} & 1 \\ 1/2 & -1 \end{bmatrix}$ but $\begin{bmatrix} 4 & 1 \\ 6 & 1 \end{bmatrix} \neq \begin{bmatrix} 4 & 0 \\ 6 & 1 \end{bmatrix}$

or $\begin{bmatrix} 4 & 1 \\ 6 & 1 \end{bmatrix} \neq \begin{bmatrix} 4 & 1 & 2 \\ 6 & 1 & 2 \end{bmatrix}$

* Adding/subtracting matrices: can only be done with matrices that have the same # of rows & columns.

ex.3) $A = \begin{bmatrix} 2 & 4 & 8 & -3 \\ 0 & 1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -3 & 4 & 0 & 1 \\ 6 & 8 & 2 & 0 \end{bmatrix}$

(a) Find $A+B = \begin{bmatrix} 2+(-3) & 4+4 & 8+0 & -3+1 \\ 0+6 & 1+8 & 2+2 & 3+0 \end{bmatrix}$

$= \boxed{\begin{bmatrix} -1 & 8 & 8 & -2 \\ 6 & 9 & 4 & 3 \end{bmatrix}}$

(b) Find $A-B = \begin{bmatrix} 2-(-3) & 4-4 & 8-0 & -3-1 \\ 0-6 & 1-8 & 2-2 & 3-0 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 0 & 8 & -4 \\ -6 & -7 & 0 & 3 \end{bmatrix}}$

suppose A, B, & C are m by n matrices then:

→ matrix addition is commutative

$$A+B=B+A$$

→ matrix addition is associative

$$(A+B)+C=A+(B+C)$$

ex.5) operations w/ matrices:

$A=\begin{bmatrix}3&1&5\\-2&0&6\end{bmatrix}$ $B=\begin{bmatrix}4&1&0\\8&1&-3\end{bmatrix}$ $C=\begin{bmatrix}9&0\\-3&6\end{bmatrix}$

Find:

(a) $4A=4\begin{bmatrix}3&1&5\\-2&0&6\end{bmatrix}=\begin{bmatrix}4\cdot3&4\cdot1&4\cdot5\\4\cdot(-2)&4\cdot0&4\cdot6\end{bmatrix}=\boxed{\begin{bmatrix}12&4&20\\-8&0&24\end{bmatrix}}$

(b) $\frac{1}{3}C=\frac{1}{3}\begin{bmatrix}9&0\\-3&6\end{bmatrix}=\begin{bmatrix}\frac{1}{3}(9)&\frac{1}{3}(0)\\\frac{1}{3}(-3)&\frac{1}{3}(6)\end{bmatrix}=\boxed{\begin{bmatrix}3&0\\-1&2\end{bmatrix}}$

(c) $3A-2B=3\begin{bmatrix}3&1&5\\-2&0&6\end{bmatrix}-2\begin{bmatrix}4&1&0\\8&1&-3\end{bmatrix}$

$=\begin{bmatrix}9&3&15\\-6&0&18\end{bmatrix}-\begin{bmatrix}8&2&0\\16&2&-6\end{bmatrix}=\boxed{\begin{bmatrix}1&1&15\\-22&-2&24\end{bmatrix}}$

DEFINITIONS:

Row Vector R is a 1 by n matrix:

$R=[r_1\ r_2\ \cdots\ r_n]$

A column vector C is an n by 1 matrix

$$C=\begin{bmatrix}c_1\\c_2\\\vdots\\c_n\end{bmatrix}$$

* The product RC of R times C is defined by the #:

$$RC = [r_1\ r_2 \cdots r_n] \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = r_1c_1 + r_2c_2 + \cdots + r_nc_n$$

NOTICE: a row vector & column vector can only be multiplied if the have the same # of entries.

ex. 6) IF $R = [3\ \ {-5}\ \ 2]$ & $C = \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix}$

$$RC = [3\ \ {-5}\ \ 2] \begin{bmatrix} 3 \\ 4 \\ -5 \end{bmatrix} = 3(3) + (-5)4 + 2(-5) = \boxed{-21}$$

* For the product of 2 matrices:

A	B
m by r	r by n

must be same # for AB to be found | defined

AB is m by n

ex.) if A is a 3x6 & B is a 6x1
we can multiply AB since A has 6 columns & B has 6 rows. Then AB will be a 3x1 matrix.

ex.8) Find AB if

$$A = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \quad \& \quad B = \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

$$* \quad 2 \times \underbrace{3 \qquad 3}_{\text{match } \checkmark} \times 4$$

AB will be a 2X4

$$AB = \begin{bmatrix} 2 & 4 & -1 \\ 5 & 8 & 0 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 & 4 \\ 4 & 8 & 0 & 6 \\ -3 & 1 & -2 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} \begin{bmatrix} 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} & \begin{bmatrix} 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix} & \begin{bmatrix} 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 2 & 4 & -1 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix} \\ \begin{bmatrix} 5 & 8 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix} & \begin{bmatrix} 5 & 8 & 0 \end{bmatrix}\begin{bmatrix} 5 \\ 8 \\ 1 \end{bmatrix} & \begin{bmatrix} 5 & 8 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} & \begin{bmatrix} 5 & 8 & 0 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ -1 \end{bmatrix} \end{bmatrix}$$

$$AB = \begin{bmatrix} 2\cdot2+4\cdot4+(-1)(-3) & 2\cdot5+4\cdot8+(-1)(1) & 2\cdot1+4\cdot0+(-1)(-2) & 2\cdot4+4\cdot6+(-1)(-1) \\ 5\cdot2+8\cdot4+0(-3) & 5\cdot5+8\cdot8+0\cdot1 & 5\cdot1+8\cdot0+0(-2) & 5\cdot4+8\cdot6+0(-1) \end{bmatrix}$$

$$\boxed{AB = \begin{bmatrix} 23 & 41 & 4 & 33 \\ 42 & 89 & 5 & 68 \end{bmatrix}}$$

ex.9) multiplying matrices

if: $A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}$

find: (a) $AB = \underset{2\times3}{\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}} \underset{3\times2}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}} = \boxed{\underset{2\times2}{\begin{bmatrix} 13 & 7 \\ -1 & -1 \end{bmatrix}}}$

(b) $BA = \underset{3\times2}{\begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}} \underset{2\times3}{\begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 0 \end{bmatrix}} = \boxed{\underset{3\times3}{\begin{bmatrix} 2 & 1 & 3 \\ 5 & 1 & 6 \\ 8 & 1 & 9 \end{bmatrix}}}$

*Matrix mult. is NOT commutative $AB \neq BA$
but it is true that:

$$A(BC) = (AB)C$$

and

$$A(B+C) = AB + AC$$

*Identity Matrix: Matrix w/ diagonals 1 and all other #'s Ø.

$\underset{\substack{\uparrow \\ 2\times2}}{I_2} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\underset{\substack{\uparrow \\ 3\times3}}{I_3} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

ex.11) $A = \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ find:

(a) $I_2A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} -1 & 2 & 0 \\ 0 & 1 & 3 \end{bmatrix} = A}$

*Identity Property :

If A is an m by n matrix then:
$$I_m A = A \quad \text{and} \quad A I_n = A$$

If A is a square :
$$A I_n = I_n A = A$$

*<u>Inverse</u> : Let A be a square matrix (nxn), if there exists an nxn A^{-1} for which :

$$AA^{-1} = A^{-1}A = I_n$$

then A^{-1} is called the inverse of A.

<u>Finding inverse for a Matrix</u>

ex.13)
$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix}$$

<u>Step 1</u> : Form the matrix $[A|I_3]$ → $[A|I_3] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right]$

<u>Step 2</u> : Transform $[A|I_3]$ into reduced row echelon form

ex. $\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & a & b & c \\ 0 & 1 & 0 & d & e & f \\ 0 & 0 & 1 & g & h & i \end{array}\right]$

on back →

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ -1 & 3 & 4 & 0 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = r_1 + r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 4 & 4 & 1 & 1 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \rightarrow$$

$$\xrightarrow{R_2 = \frac{1}{4} r_2} \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1/4 & 1/4 & 0 \\ 0 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3 = -4r_2 + r_3]{R_1 = -1r_2 + r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3/4 & -1/4 & 0 \\ 0 & 1 & 1 & 1/4 & 1/4 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{R_3 = -1r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 3/4 & -1/4 & 0 \\ 0 & 1 & 1 & 1/4 & 1/4 & 0 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right] \xrightarrow[R_2 = -1r_3 + r_2]{R_1 = r_3 + r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/4 & 3/4 & -1 \\ 0 & 1 & 0 & -3/4 & -3/4 & 1 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$$

inverse!!

so $\boxed{A^{-1} = \begin{bmatrix} 7/4 & 3/4 & -1 \\ -3/4 & -3/4 & 1 \\ 1 & 1 & -1 \end{bmatrix}}$

ex.14) show $A = \begin{bmatrix} 4 & 6 \\ 2 & 3 \end{bmatrix}$ has no inverse

$$\left[\begin{array}{cc|cc} 4 & 6 & 1 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_1 = \frac{1}{4} r_1} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/4 & 0 \\ 2 & 3 & 0 & 1 \end{array}\right] \xrightarrow{R_2 = -2r_1 + r_2} \left[\begin{array}{cc|cc} 1 & 3/2 & 1/4 & 0 \\ 0 & 0 & -1/2 & 1 \end{array}\right]$$

since not the identity matrix = not inverse

*HINT! if the determinant of a matrix is 0, the matrix is singular

↑ means no inverse

ex.15) using matrices to solve a system of linear equations with inverses.

$$\begin{cases} x + y = 3 \\ -x + 3y + 4z = -3 \\ 4y + 3z = 2 \end{cases}$$

Let:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 3 & 4 \\ 0 & 4 & 3 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}$$

so we could write the system as:

$$AX = B$$

if I want to solve for X

$$\cancel{A^{-1}(A)}X = B(A^{-1})$$

$$X = A^{-1}B$$

so

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1}B = \underset{3\times3}{\begin{bmatrix} 7/4 & 3/4 & -1 \\ -3/4 & -3/4 & 1 \\ 1 & 1 & -1 \end{bmatrix}} \underset{3\times1}{\begin{bmatrix} 3 \\ -3 \\ 2 \end{bmatrix}} = \underset{3\times1}{\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}}$$

so $\boxed{x = 1 \quad y = 2 \quad z = -2 \quad \text{or} \quad (1, 2, -2)}$