12.3 Determinants

$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$ * if a, b, c, d, are 4 real #'s D is called a <u>2x2 determinant</u>

where its value is the number $ad-bc$.

bc

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

ad

minus

ex. 1) Evaluate: $\begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix} = 3(1)-6(-2) = 3-(-12) = \boxed{15}$

<u>Theorem</u>: Cramer's Rule for 2 equations containing 2 variables

The solution to: $\begin{cases} ax+by=s \\ cx+dy=t \end{cases}$

is given by:

$$x = \frac{\begin{vmatrix} s & b \\ t & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} a & s \\ c & t \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}}$$

provided that:

$$D = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc \neq 0$$

cramer's rule pattern

$$\begin{cases} ax+by = s \\ cx+dy = t \end{cases} \qquad D = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

$$Dx = \begin{vmatrix} s & b \\ t & d \end{vmatrix} \qquad Dy = \begin{vmatrix} a & s \\ c & t \end{vmatrix}$$

so $x = \frac{Dx}{D}$ $\quad y = \frac{Dy}{D}$ if $D \neq 0$

ex.2) use cramer's Rule to solve:

$$\begin{cases} 3x - 2y = 4 \\ 6x + y = 13 \end{cases}$$

Find D First: $D = \begin{vmatrix} 3 & -2 \\ 6 & 1 \end{vmatrix} = (3)(1) - (6)(-2) = 15$

$D = 15$

Find Dx & Dy: $Dx = \begin{vmatrix} 4 & -2 \\ 13 & 1 \end{vmatrix}$ $\quad Dy = \begin{vmatrix} 3 & 4 \\ 6 & 13 \end{vmatrix}$

$Dx = 4(1) - 13(-2) = 30$ $\quad Dy = 3(13) - 6(4)$

$Dx = 30$ $\quad Dy = 15$

Find x & y: $x = \frac{Dx}{D} = \frac{30}{15} = 2$ $\quad y = \frac{Dy}{D} = \frac{15}{15} = 1$

so $\boxed{x = 2 \text{ and } y = 1 \text{ or } (2,1)}$

* Evaluate a 3X3 Determinant

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

found by removing the row & column containing a_{11}

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

The 2X2 determinants are called <u>minors</u>

ex. 3) Find minors:

$$A = \begin{vmatrix} 2 & -1 & 3 \\ -2 & 5 & 1 \\ 0 & 6 & -9 \end{vmatrix}$$

find: (a) M_{12} ← row 1 column 2

$$\begin{vmatrix} \cancel{2} & \cancel{-1} & \cancel{3} \\ -2 & \cancel{5} & 1 \\ 0 & \cancel{6} & -9 \end{vmatrix} \rightarrow \begin{vmatrix} -2 & 1 \\ 0 & -9 \end{vmatrix}$$

$$M_{12} = (-2)(-9) - (0)(1) = \boxed{18}$$

(b) M_{23} ← row 2 column 3

$$\begin{vmatrix} 2 & -1 & \cancel{3} \\ \cancel{-2} & \cancel{5} & \cancel{1} \\ 0 & 6 & \cancel{-9} \end{vmatrix} \rightarrow \begin{vmatrix} 2 & -1 \\ 0 & 6 \end{vmatrix}$$

$$M_{23} = (2)(6) - (0)(-1) = \boxed{12}$$

DO NOT NEED TO KNOW

* For an $n \times n$ determinant, the cofactor of entry a_{ij} denoted by A_{ij} is given by:

$$A_{ij} = (-1)^{i+j} M_{ij}$$

where M_{ij} is the minor of entry a_{ij}

ex.4) Evaluate a 3X3:

$$\begin{vmatrix} 3 & 0 & -1 \\ 4 & 6 & 2 \\ 8 & -2 & 3 \end{vmatrix}$$

$$3\begin{vmatrix} 6 & 2 \\ -2 & 3 \end{vmatrix} - 0\begin{vmatrix} 4 & 2 \\ 8 & 3 \end{vmatrix} + (-1)\begin{vmatrix} 4 & 6 \\ 8 & -2 \end{vmatrix}$$

$$3(18-(-4)) - 0\cancel{(12-16)} + (-1)(-8-48)$$

$$3(22) + (-1)(-56)$$

$$66 + 56 = \boxed{122}$$

* Theorem: Cramer's Rule for 3 equation, 3 variable

For:

$$\begin{cases} a_{11}x + a_{12}y + a_{13}z = c_1 \\ a_{21}x + a_{22}y + a_{23}z = c_2 \\ a_{31}x + a_{32}y + a_{33}z = c_3 \end{cases}$$

&:

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \neq 0$$

then: $x = \frac{D_x}{D}$ $y = \frac{D_y}{D}$ $z = \frac{D_z}{D}$ where:

$$D_x = \begin{vmatrix} c_1 & a_{12} & a_{13} \\ c_2 & a_{22} & a_{23} \\ c_3 & a_{32} & a_{33} \end{vmatrix} \quad D_y = \begin{vmatrix} a_{11} & c_1 & a_{13} \\ a_{21} & c_2 & a_{23} \\ a_{31} & c_3 & a_{33} \end{vmatrix} \quad D_z = \begin{vmatrix} a_{11} & a_{12} & c_1 \\ a_{21} & a_{22} & c_2 \\ a_{31} & a_{32} & c_3 \end{vmatrix}$$

ex. 5) use cramer's rule to solve:

$$\begin{cases} 2x + y - z = 3 \\ -x + 2y + 4z = -3 \\ x - 2y - 3z = 4 \end{cases}$$

$$D = \begin{vmatrix} 2 & 1 & -1 \\ -1 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = 2\begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix}$$

$$= 2(2) - 1(-1) + (-1)(0)$$

$$= 4 + 1 = 5$$

$\underline{D = 5}$

$$D_x = \begin{vmatrix} 3 & 1 & -1 \\ -3 & 2 & 4 \\ 4 & -2 & -3 \end{vmatrix} = 3\begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix} - 1\begin{vmatrix} -3 & 4 \\ 4 & -3 \end{vmatrix} + (-1)\begin{vmatrix} -3 & 2 \\ 4 & -2 \end{vmatrix}$$

$$= 3(2) - 1(-7) + (-1)(-2) = 15$$

$\underline{D_x = 15}$

$$D_y = \begin{vmatrix} 2 & 3 & -1 \\ -1 & -3 & 4 \\ 1 & 4 & -3 \end{vmatrix} = 2\begin{vmatrix} -3 & 4 \\ 4 & -3 \end{vmatrix} - 3\begin{vmatrix} -1 & 4 \\ 1 & -3 \end{vmatrix} + (-1)\begin{vmatrix} -1 & -3 \\ 1 & 4 \end{vmatrix} = -10$$

$\underline{D_y = -10}$

$$D_z = \begin{vmatrix} 2 & 1 & 3 \\ -1 & 2 & -3 \\ 1 & -2 & 4 \end{vmatrix} = 2\begin{vmatrix} 2 & -3 \\ -2 & 4 \end{vmatrix} - 1\begin{vmatrix} -1 & -3 \\ 1 & 4 \end{vmatrix} + 3\begin{vmatrix} -1 & 2 \\ 1 & -2 \end{vmatrix} = 5$$

$D_z = 5$

so: $x = \frac{15}{5}$ $\quad y = \frac{-10}{5}$ $\quad z = \frac{5}{5}$

$\boxed{x = 3 \quad y = -2 \quad z = 1 \quad \text{or } (3, -2, 1)}$