12.2 Systems of Linear Equations: Matrices

consider the following system of linear equation:

$$\begin{cases} x + 4y = 14 \\ 3x - 2y = 0 \end{cases}$$

we could write it as:

$$\left[\begin{array}{cc|c} 1 & 4 & 14 \\ 3 & -2 & 0 \end{array}\right]$$

coefficients with x

coefficients with y

=

constants

* <u>Matrix</u>: a rectangular array of #s

$$\begin{array}{rc} & \begin{array}{cccccc} \text{column 1} & \text{column 2} & & \text{column } j & & \text{column } n \end{array} \\ \begin{array}{r} \text{Row 1} \\ \text{Row 2} \\ \\ \text{Row } i \\ \\ \text{Row } m \end{array} & \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \end{array}$$

a_{mn} ← column #

row #

* matrix used to represent a system of linear equations is called an <u>augmented matrix</u>.

ex.1) write the matrix for:

(a) $\begin{cases} 3x-4y=-6 \\ 2x-3y=-5 \end{cases}$ $\boxed{\left[\begin{array}{cc|c} 3 & -4 & -6 \\ 2 & -3 & -5 \end{array}\right]}$

(b) $\begin{cases} 2x-y+z=0 \\ x+z-1=0 \\ x+2y-8=0 \end{cases} \xrightarrow{\text{rewrite}} \begin{cases} 2x-1y+1z=0 \\ 1x+0y+1z=1 \\ 1x+2y+0z=8 \end{cases}$

$$\boxed{\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 0 & 8 \end{array}\right]}$$

ex.2) write system of linear equation for:

(a) $\left[\begin{array}{cc|c} 5 & 2 & 13 \\ -3 & 1 & -10 \end{array}\right]$ $\boxed{\begin{cases} 5x+2y=13 \\ -3x+y=-10 \end{cases}}$

(b) $\left[\begin{array}{ccc|c} 3 & -1 & -1 & 7 \\ 2 & 0 & 2 & 8 \\ 0 & 1 & 1 & 0 \end{array}\right]$ $\boxed{\begin{cases} 3x-y-z=7 \\ 2x+2z=8 \\ y+z=0 \end{cases}}$

* Row Operations:
1. Interchange any two rows
2. Replace a row by a nonzero multiple of that row
3. Replace a row by the sum of that row and a constant nonzero multiple of some other row.

ex.3) Apply the row operation $R_2 = -3r_1 + r_2$ to:

R_2 represents new row 2; r_1 row 1 original #s; r_2 row 2 original #s

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 3 & -5 & 9 \end{array}\right]$$

$$\downarrow$$

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ -3(1)+3 & -3(-2)+(-5) & -3(2)+9 \end{array}\right] = \boxed{\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]}$$

ex.4) Finding a Row operation:

$$\left[\begin{array}{cc|c} 1 & -2 & 2 \\ 0 & 1 & 3 \end{array}\right]$$ to make 0 in row1 column 2

since there is a 1 we can use it to get rid of -2

$R_1 = 2r_2 + r_1$

$$\left[\begin{array}{cc|c} 2(0)+1 & 2(1)+-2 & 2(3)+2 \\ 0 & 1 & 3 \end{array}\right] = \boxed{\left[\begin{array}{cc|c} 1 & 0 & 8 \\ 0 & 1 & 3 \end{array}\right]}$$

so $\boxed{\begin{array}{l} x = 8 \\ y = 3 \end{array}}$ *

* Row echelon form : when

1. Entry in row1 column1 is a 1 & 0's below it.
2. 1st nonzero entry in each row after the 1st row is a 1, only 0's appear below.
3. Any rows that contain all 0's to the left of the vertical bar appear at the bottom.

ex. $$\left[\begin{array}{ccc|c} 1 & a & b & d \\ 0 & 1 & c & e \\ 0 & 0 & 1 & f \end{array}\right]$$

ex. 5) solve: $\begin{cases} 2x+2y = 6 \\ x+y+z = 1 \\ 3x+4y-z = 13 \end{cases}$

Step 1: write augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 0 & 6 \\ 1 & 1 & 1 & 1 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

switch row 1 & 2

Step 2: get a 1 in row 1 column 1

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & 2 & 0 & 6 \\ 3 & 4 & -1 & 13 \end{array}\right]$$

$R_2 = -2r_1 + r_2$

$R_3 = -3r_1 + r_3$

Step 3: get 0's under the 1 in column 1

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 1 & -4 & 10 \end{array}\right]$$

Step 4: we want row 2 column 2 to be a 1 & 0's under

switch row 2 & 3

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & -2 & 4 \end{array}\right]$$

Step 5: get a 1 in row 3 column 3

$R_3 = -\frac{1}{2}r_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & -4 & 10 \\ 0 & 0 & 1 & -2 \end{array}\right]$$

Step 6: solve

$x + y + z = 1$

$y + (-4)z = 10$

$\boxed{z = -2}$

$y + (-4)(-2) = 10$

$\boxed{y = 2}$

$x + 2 + -2 = 1$ so $\boxed{x = 1}$

ex.6) solve
$$\begin{cases} x - y + z = 8 \\ 2x + 3y - z = -2 \\ 3x - 2y - 9z = 9 \end{cases}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 2 & 3 & -1 & -2 \\ 3 & -2 & -9 & 9 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 5 & -3 & -18 \\ 0 & 1 & -12 & -15 \end{array}\right]$$

$R_2 = -2r_1 + r_2$
$R_3 = -3r_1 + r_3$

$$\xrightarrow[\text{switch } r_2 \,\&\, r_3]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 5 & -3 & -18 \end{array}\right] \xrightarrow[R_3 = -5r_2 + r_3]{} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 57 & 57 \end{array}\right]$$

$$\rightarrow R_3 = \frac{1}{57} r_3 \quad \left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right] \text{ so } \begin{cases} x - y + z = 8 \\ y - 12z = -15 \\ z = 1 \end{cases}$$

answer $\boxed{(4, -3, 1)}$

*could write in reduced row echelon form!

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 8 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[R_1 = r_2 + r_1]{} \left[\begin{array}{ccc|c} 1 & 0 & -11 & -7 \\ 0 & 1 & -12 & -15 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[\substack{R_1 = 11r_3 + r_1 \\ R_2 = 12r_3 + r_2}]{}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 \end{array}\right] \leftarrow \text{answers} \quad \boxed{\begin{array}{l} x = 4 \\ y = -3 \quad \text{or } (4, -3, 1) \\ z = 1 \end{array}}$$

ex.7) Solve $\begin{cases} 6x - y - z = 4 \\ -12x + 2y + 2z = -8 \\ 5x + y - z = 3 \end{cases}$

$$\left[\begin{array}{ccc|c} 6 & -1 & -1 & 4 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right] \xrightarrow{R_1 = -1r_3 + r_1} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ -12 & 2 & 2 & -8 \\ 5 & 1 & -1 & 3 \end{array}\right]$$

$$\xrightarrow[R_3 = -5r_1 + r_3]{R_2 = 12r_1 + r_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & -22 & 2 & 4 \\ 0 & 11 & -1 & -2 \end{array}\right] \xrightarrow{R_2 = -\frac{1}{22}r_2} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1/11 & -2/11 \\ 0 & 11 & -1 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 = -11r_2 + r_3} \left[\begin{array}{ccc|c} 1 & -2 & 0 & 1 \\ 0 & 1 & -1/11 & -2/11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

means infinitely many solutions

so $x - 2y = 1$

$y - \frac{1}{11}z = -\frac{2}{11}$

$y = -\frac{2}{11} + \frac{1}{11}z$

$x = 1 + 2\left(-\frac{2}{11} + \frac{1}{11}z\right)$

$$\boxed{y = -\frac{2}{11} + \frac{1}{11}z \quad x = \frac{7}{11} + \frac{2}{11}z}$$

z can be any real #

ex.8) $\begin{cases} x + y + z = 6 \\ 2x - y - z = 3 \\ x + 2y + 2z = 0 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 2 & -1 & -1 & 3 \\ 1 & 2 & 2 & 0 \end{array}\right] \xrightarrow[R_3 = -1r_1 + r_3]{R_2 = -2r_1 + r_2} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & -3 & -3 & -9 \\ 0 & 1 & 1 & -6 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & -3 & -3 & -9 \end{array}\right]$$

$$\xrightarrow{R_3 = 3r_2 + r_3} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 1 & -6 \\ 0 & 0 & 0 & -27 \end{array}\right] \leftarrow 0x + 0y + 0z = -27$$

NO solution (inconsistent)

* inconsistent – no solution

* consistent – 1 solution or infinitely many solutions

ex.9) $\begin{cases} x - 2y + z = 0 \\ 2x + 2y - 3z = -3 \\ y - z = -1 \\ -x + 4y + 2z = 13 \end{cases}$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 2 & 2 & -3 & -3 \\ 0 & 1 & -1 & -1 \\ -1 & 4 & 2 & 13 \end{array}\right] \xrightarrow[R_4 = r_1 + r_4]{R_2 = -2r_1 + r_2}$$

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 6 & -5 & -3 \\ 0 & 1 & -1 & -1 \\ 0 & 2 & 3 & 13 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 6 & -5 & -3 \\ 0 & 2 & 3 & 13 \end{array}\right] \xrightarrow[R_4 = -2r_2 + r_4]{R_3 = -6r_2 + r_3} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

$$\xrightarrow{R_4 = -5r_3 + r_4} \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = 3$

$y - z = -1$

$x - 2y + z = 0$

so $\boxed{(1, 2, 3)}$